MS&E 213 / CS 269O: Appendix - Chapter A - Norms*

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Here we review some basic facts about norms we use throughout the course

1 Norms

We begin by recalling some basic facts about norms.

Definition 1. $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is called a *norm* if the following hold

- *(absolute homogeneity)* $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$
- *(triangle inequality)* $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$
- *(distinguishing 0)* if $\|x\| = 0$ then $x = \vec{0}$

If the last condition does not hold we say that $\|\cdot\|$ is a semi-norm.

We note some immediate consequence of this definition.

Lemma 2. *If $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is a norm then $\|\vec{0}\| = 0$ and $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$.*

Proof. The first fact follows from the fact that absolute homogeneity implies

$$\|\vec{0}\| = \|0 \cdot \vec{0}\| = |0| \cdot \|\vec{0}\| = 0$$

and the second follows from the fact that triangle inequality and absolute homogeneity imply

$$\|0\| = \|x - x\| \leq \|x\| + \| - x\| = 2 \cdot \|x\| \,.$$

□

Now for every norm there is a natural induced dual norm given as follows.

Definition 3. For a norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ we define the dual norm $\|\cdot\|_* : \mathbb{R}^n \to \mathbb{R}$ for all $x \in \mathbb{R}^n$ by

$$\|x\|_* = \max_{\|y\| \leq 1} y^\top x \,.$$

*These notes are a work in progress. They are not necessarily a subset or superset of the in-class material and there may also be occasional *TODO* comments which demarcate material I am thinking of adding in the future. These notes will converge to a superset of the class material that is TODO-free. Your feedback is welcome and highly encouraged. If anything is unclear, you find a bug or typo, or if you would find it particularly helpful for anything to be expanded upon, please do not hesitate to post a question on the discussion board or contact me directly at sidford@stanford.edu.

It is not too hard to show that the dual norm is always a norm.

Lemma 4. *For a norm* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ *it's dual norm* $\|\cdot\|_*$ *is a norm.*

Proof. Note that for all $x \in \mathbb{R}^n$ since for $\widehat{x} = \frac{1}{\|x\|} x$ it is the case $\|\widehat{x}\| = 1$ we have

$$\|x\|_* = \max_{\|z\| \leq 1} z^\top x \geq \widehat{x}^\top x = \frac{\|x\|_2^2}{\|x\|} \, .$$

Consequently $\|x\|_* \geq 0$ for all $x \neq 0$ and clearly $\|0\|_* = 0$. Using this we see that

$$\|\alpha \cdot x\|_* = \max_{\|z\| \leq 1} \alpha \cdot z^\top x = \max_{\|z\| \leq 1} |\alpha| \cdot z^\top x = |\alpha| \cdot \|x\|_*$$

where we used that $\max_{\|z\| \leq} z^\top x = \max_{\|z\| \leq 1} z^\top(-x)$ since if $\|z\| \leq 1$ then $\|-z\| \leq 1$. The last property follows from

$$\|x + y\|_* = \max_{\|z\| \leq 1} z^\top(x + y) \leq \max_{\|z\| \leq 1} z^\top x + \max_{\|w\| \leq 1} z^\top y = \|x\|_* + \|y\|_* \, .$$

□

It is also easy to show that Cauchy Schwarz then holds in any norm.

Lemma 5 (Cauchy Schwarz)**.** *For a norm* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ *it's dual norm* $\|\cdot\|_*$ *and all* $x, y \in \mathbb{R}^n$ *we have*

$$\left| x^\top y \right| \leq \|x\| \cdot \|y\|_* \, .$$

Proof. If either $x = 0$ or $y = 0$ then the claim is trivial. Otherwise we have

$$x^\top y = \|x\| \left(\frac{x}{\|x\|} \right)^\top y \leq \|x\| \cdot \max_{\|z\| \leq 1} z^\top y = \|x\| \cdot \|y\|_* \, .$$

Applying the same to $(-x)^\top y$ we have the desired result. □

The norm also shows up naturally in solving unconstrained minimization problems involving the norm, for example, we can show the following:

Lemma 6. *For any norm* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ *and* $x \in \mathbb{R}^n$ *and* $\alpha > 0$ *we have that for*

$$\min_y x^\top y + \frac{\alpha}{2} \|y\|^2 = -\frac{1}{2\alpha} \|x\|_*^2$$

Proof. HW. □

References